[11-03-22-T12]
Notes on FTC

The antiderivative of a function

The symbol $\int f(x)\, dx$ represents the antiderivative of the function *f*; that is, it represents the family of functions each of whose derivative is *f*. If a point on the antiderivative of *f* is known, then the antiderivative of *f* is a single function. The symbol $\int \ldots dx$ is a linear operator that produces the antiderivative of . . . . The symbol $\int f(x)\, dx$ is often called the indefinite integral of the function *f*.

The definite integral

The symbol $\int_a^b f(x)\, dx$ is a *number*. It may be thought of as the signed area of the region bounded by *f*, the x-axis, and the lines $x = a,\ x = b$. The definition of $\int_a^b f(x)\, dx$ says that it is $\lim_{|P|\to 0} \sum_{i=1}^{n} f(x_i^*)\,(\Delta_i)$ which is the limit of a Riemann sum as the norm of thge partition goes to zero. In general, it is difficult to determine the numeric value of $\int_a^b f(x)\, dx$. The symbol $\int_a^b f(x)\, dx$ is called the definite integral.

The Fundamental Theorem of Calculus

First part

$\int_a^b f(t)\, dt$ is a particular number. As noted, it may be thought of as the signed area of a region. But, the symbol $\int_a^x f(t)\, dt$ is a function whose argument in this case is x. It may be thought of as the signed area of the region bounded by *f*, the t-axis, and the lines $t = a,\ t = x$. We wonder, of course, about how to find the derivative, which may be thought of as the instantaneous change in $A(x)$ at x, of the function $A(x) = \int_a^x f(t)\, dt$. The answer is provided by the first part of the FTC (Fundamental Theorem of Calculus) which says that if *f* is continuous on $[a, b]$, and $a \le x \le b$, then $\frac{d}{dx} A(x) = f(x)$. That is, the instantaneous rate of change in the function $\int_a^x f(t)\, dt$ at x is equal to the value of the function *f* at x. Do note that if $A(x) = \int_a^u f(t)\, dt$ where u is a differentiable function of x, then the Chain Rule applies - just as when taking the derivative of any composite function. Thus, $\frac{d}{dx}\left(\int_a^u f(t)\, dt\right) = f(u) \cdot \frac{du}{dx}$.

Second part

As noted, the number $\int_a^b f(x)\, dx$ is the limit of a Riemann sum and as such it is a number that might be quite difficult or at best cumbersome to compute. The second part of the FTC is supposed to make easy the computation of this number. The second part of the FTC says that if *f* is continuous on $[a, b]$, then the number $\int_a^b f(x)\, dx$ is equal to the difference of an antiderivative of *f* evaluated at a and at b; that is, if the antiderivative of *f* is F, then the number $\int_a^b f(x)\, dx$ is equal to $F(b) - F(a)$. In general, it is difficult to find the an antiderivative of *f*, but there are several strategies that work for a large number of cases.